Security Amplification for the Composition of Block Ciphers: Simpler Proofs and New Results

B. Cogliati [1] J. Patarin [1] Y. Seurin [2]

[1]University of Versailles, FRANCE

[2]ANSSI, FRANCE

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We consider the general problem of *security amplification* for blockciphers:

Problem

Given two or more blockciphers $E, F \ldots$ does the composition $E \circ F \cdots$ offer better security than each component?

- widely studied problem, lots of results in different models (computational model, information-theoretic model, ideal cipher model,...)
- we focus here on the information-theoretic model (computationally unbouded adversaries)
- starting point of our work: the famous "Two weak make one strong" theorem

Theorem (2W1S theorem)

If E and F are (q, ϵ) secure against chosen plaintext non-adaptive (NCPA) adversaries, then $F^{-1} \circ E$ is $(q, 2\epsilon)$-secure against chosen plaintext and ciphertext (CCA) adversaries

Previous proof was long and complex [Mau02, MPR07, JÖS12].

Our results in short

- we give a surpringly simple proof of the 2W1S theorem,
- we extend it to any number of rounds.

The distinguishing advantage of an adversary

Fix a block cipher E with key space $\mathcal{K}$ and message space $\mathcal{M}$. A distinguisher D is an algorithm with oracle access to a permutation F which outputs a bit D^F.

His advantage is

$$\left|\Pr\left[K \leftarrow_{\$} \mathcal{K} : D^{E_K} = 1\right] - \Pr\left[P \leftarrow_{\$} \mathrm{Perm}(\mathcal{M}) : D^P = 1\right]\right|.$$

$\mathbf{Adv}_E^{\mathrm{cca}}(q)$: maximum advantage when D is limited to q queries.
$\mathbf{Adv}_E^{\mathrm{ncpa}}(q)$: maximum advantage when D is limited to q non-adaptive forward queries.
$\mathbf{Adv}_E^{\mathrm{cpa}}(q)$: maximum advantage when D is limited to q adaptive forward queries.

A preview of the results

Composing block cipher with independant keys improves security:

- the gain for ncpa and cpa security is geometric,
- to achieve the same level of cca security from ncpa-secure block ciphers, one must double the length of the cascade.

We show that only one round must be added to get roughly the same level of cca security.

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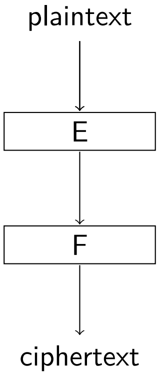
Security amplification

2 types of security amplification:

- ϵ-amplification,
- class amplification.

Example of $\epsilon-$amplification (from [Vau98, Vau99])



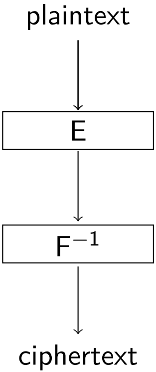
E is $(q, \epsilon_E)-$NCPA (resp. CPA) secure.

F is $(q, \epsilon_F)-$NCPA (resp. CPA) secure.

$\rightarrow$ We get a $(q, 2\epsilon_E\epsilon_F)-$NCPA (resp CPA) secure blockcipher.

Example of class amplification

"Two weak make one strong" (TW1S) theorem



E is $(q, \epsilon_E)-$NCPA secure.

F is $(q, \epsilon_F)-$NCPA secure.

$\rightarrow$ We get a $(q, \epsilon_E + \epsilon_F)-$CCA secure blockcipher.

Example of class amplification

"Two weak make one strong" (TW1S) theorem

This theorem is used in several proofs [MRS09, HR10, LPS12, LS14].

However its proof relies on three articles : [Mau02], [MPR07] and [JÖS12].

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Stastistical distance

Let μ and ν be 2 probability distributions on a finite event space Ω. The stastistical distance between μ and ν is:

$$\begin{aligned}\|\mu - \nu\| &= \frac{1}{2}\sum_{\omega \in \Omega} |\mu(\omega) - \nu(\omega)| \\ &= \sum_{\substack{\omega \in \Omega \\ \mu(\omega) > \nu(\omega)}} (\mu(\omega) - \nu(\omega))\end{aligned}$$

Some notations

Let x, y be $q-$tuples of pairwise distinct messages from $\mathcal{M}$ and E a block cipher with message space $\mathcal{M}$. Then

- $\mathrm{p}_E(x, y)$ is the probability, over the choice of the key, that E outputs y with input x,
- $\mathrm{p}_{E,x}$ is the probability distribution of the outputs of E when the input x is fixed,
- $\mathrm{p}^* = \frac{1}{|\mathcal{M}|(|\mathcal{M}|-1)\ldots(|\mathcal{M}|-q+1)}$.

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Fundamental results of the H-coefficient method

Lemma

Let E be a block cipher with message space $\mathcal{M}$. Denote $(\mathcal{M})_q$ the set of $q-$tuples of pairwise distinct messages of $\mathcal{M}$. Then

$$\mathbf{Adv}_E^{\mathrm{ncpa}}(q) = \max_{x \in (\mathcal{M})_q} \|\mathsf{p}_{E,x} - \mathsf{p}^*\|.$$

Fundamental results of the H-coefficient method

Lemma

Let E be a block cipher with message space $\mathcal{M}$. Assume that there exists ϵ such that for any $q-$tuples $x, y \in (\mathcal{M})_q$, one has

$$\mathrm{p}_E(x, y) \geq (1 - \epsilon)\mathrm{p}^*.$$

Then

$$\mathbf{Adv}_E^{\mathrm{cca}}(q) \leq \epsilon.$$

A proof of the 2W1S theorem

Let E and F be two block ciphers with the same message space $\mathcal{M}$ and respective key spaces $\mathcal{K}_E$ and $\mathcal{K}_F$. Let $x, y \in (\mathcal{M})_q$.

First step: a surprisingly simple and useful formula:

$$\mathrm{p}_{F^{-1}\circ E}(x,y) = \mathrm{p}^* + \sum_{z\in(\mathcal{M})_q} (\mathrm{p}_E(x,z) - \mathrm{p}^*)(\mathrm{p}_F(y,z) - \mathrm{p}^*)$$

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A proof of the 2W1S theorem

$$\begin{aligned} \mathrm{p}_{F^{-1}\circ E}(x,y) \geq{} & \mathrm{p}^* + \sum_{\substack{z\in(\mathcal{M})_q \\ \mathrm{p}_E(x,z)>\mathrm{p}^* \\ \mathrm{p}_F(y,z)<\mathrm{p}^*}} \underbrace{(\mathrm{p}_E(x,z)-\mathrm{p}^*)}_{>0}\underbrace{(\mathrm{p}_F(y,z)-\mathrm{p}^*)}_{\geq -\mathrm{p}^*} \\ & + \sum_{\substack{z\in(\mathcal{M})_q \\ \mathrm{p}_E(x,z)<\mathrm{p}^* \\ \mathrm{p}_F(y,z)>\mathrm{p}^*}} \underbrace{(\mathrm{p}_E(x,z)-\mathrm{p}^*)}_{\geq -\mathrm{p}^*}\underbrace{(\mathrm{p}_F(y,z)-\mathrm{p}^*)}_{>0} \end{aligned}$$

A proof of the 2W1S theorem

$$\begin{aligned}
\mathrm{p}_{F^{-1}\circ E}(x,y) \geq & \, \mathrm{p}^* - \mathrm{p}^* \underbrace{\sum_{\substack{z\in(\mathcal{M})_q \\ \mathrm{p}_E(x,z)>\mathrm{p}^*}} (\mathrm{p}_E(x,z)-\mathrm{p}^*)}_{\leq \mathbf{Adv}_E^{\mathrm{ncpa}}(q)} \\
& - \mathrm{p}^* \underbrace{\sum_{\substack{z\in(\mathcal{M})_q \\ \mathrm{p}_F(y,z)>\mathrm{p}^*}} (\mathrm{p}_F(y,z)-\mathrm{p}^*)}_{\leq \mathbf{Adv}_F^{\mathrm{ncpa}}(q)}
\end{aligned}$$

Then

$$\mathrm{p}_{F^{-1}\circ E}(x,y) \geq \mathrm{p}^*(1 - \mathbf{Adv}_E^{\mathrm{ncpa}}(q) - \mathbf{Adv}_F^{\mathrm{ncpa}}(q)).$$

Many weak make one even stronger!

Our proof scales very well to the composition of multiple block ciphers and gives:

Theorem

Let $E_1, \ldots, E_n$ be n block ciphers with the same message space $\mathcal{M}$. For any integer q, one has

$$\mathbf{Adv}^{\mathrm{cca}}_{E_n \circ \ldots \circ E_1}(q) \leq 2^{n-1} \max_{1 \leq i \leq n} \left(\prod_{j=1}^{i-1} \mathbf{Adv}^{\mathrm{ncpa}}_{E_j}(q) \times \prod_{j=i+1}^{n} \mathbf{Adv}^{\mathrm{ncpa}}_{E_j^{-1}}(q) \right).$$

Many weak make one even stronger!

Corollary

Let E be a block cipher and $q \geq 1$. Denote $\epsilon = \max\{\mathbf{Adv}_E^{\mathrm{ncpa}}(q), \mathbf{Adv}_{E^{-1}}^{\mathrm{ncpa}}(q)\}$. Then, for any integer $n \geq 1$,

$$\mathbf{Adv}_{E^n}^{\mathrm{cca}}(q) \leq (2\epsilon)^{n-1}.$$

The best we could get using previous results was:

- $\mathbf{Adv}_{E^n}^{\mathrm{cca}}(q) \leq (2\epsilon)^{n/2}$ when n is even,
- $\mathbf{Adv}_{E^n}^{\mathrm{cca}}(q) \leq (2\epsilon)^{\frac{n-1}{2}} \frac{1+2\epsilon}{2}$ when n is odd.

About the tightness of MW1S

Denote G the block cipher whose key space is the set of all permutations of $\mathcal{M}$ such that 0 lies in a circle of length 2 and F the block cipher such that:

- with probability ϵ, F is the identity function $\mathcal{I}$,
- with probability $1-\epsilon$, F is G with a uniformly random key.

Then

$$\mathbf{Adv}_{F^n}^{\mathrm{cca}}(q) \gtrapprox n \cdot \mathbf{Adv}_{F}^{\mathrm{ncpa}}(q)^{n-1}$$

when $\mathcal{M}$ is sufficiently large and ϵ sufficiently small.

Composition of 3 block ciphers

Theorem

Let E, F, G be 3 block ciphers with the same message space $\mathcal{M}$ and q be any positive integer. Denote, for any block cipher B, $\epsilon_B := \mathbf{Adv}_B^{\mathrm{ncpa}}(q)$. Then

$$\begin{aligned}\mathbf{Adv}_{G\circ F\circ E}^{\mathrm{cca}}(q) \leq & \epsilon_E\epsilon_F + \epsilon_E\epsilon_{G^{-1}} + \epsilon_{F^{-1}}\epsilon_{G^{-1}} \\ & + \min\{\epsilon_E\epsilon_F, \epsilon_E\epsilon_{G^{-1}}, \epsilon_{F^{-1}}\epsilon_{G^{-1}}\}.\end{aligned}$$

In summary,

- we give a new simple proof of the "Two weak make one strong" theorem, relying on the H-coefficients framework [Pat08],
- we extend the 2W1S theorem to any number of rounds, and show that if E and E^{-1} are (q, ϵ)-ncpa secure, then E^n is $(q, (2\epsilon)^{n-1})$-secure,
- in particular, this shows that 3 rounds are sufficient to provide *both* $\epsilon-$amplification and class amplification (E and E^{-1} (q, ϵ)-ncpa secure $=>$ E^3 is $(q, 4\epsilon^2)$-cca secure),
- our extension is tight up to some constant factor.

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Thank you!

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